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Viewing the proton through "color" filters

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Abstract. While the form factors and parton distributions provide separately the shape of the proton in coordinate and momentum spaces, a more powerful imaging of the proton structure can be obtained through quantum phase-space distributions. Here we introduce the Wigner-type quark and gluon distributions which depict a full-3D proton at every fixed Feynman momentum, like what is seen through momentum("color")-filters. After appropriate reductions, the phase-space distributions are related to the generalized parton distributions (GPDs) and transverse-momentum dependent parton distributions measurable in high-energy experiments.

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1 Introduction

In exploring the microscopic structure of matter, there are two frequently-used approaches. First, the spatial distribution of matter (or charge) in a system can be probed through elastic scattering of electrons, or photons, or neutrons, etc. The physical quantity that one measures is the elastic form (structure) factors which depend on three-momentum transfer to the system. The Fourier transformation of the form factors provides direct information on the spatial distributions. The well-known examples include the study of charge distribution in an atom and the atomic structure of a crystal. The second approach is designed to measure the population of the constituents as a function of momentum, or the momentum distribution, through knock-out scattering. Here the well-known examples include the nucleon distributions in nuclei measured through quasi-elastic electron scattering, and the distribution of atoms in a quantum liquid probed through neutron scattering. The scattering cross section sometimes depends on the reaction dynamics which must be understood before the momentum distribution can be extracted.

Both approaches are complementary, but bear similar drawbacks. The form factor measurements do not yield any information about the underlying dynamics of the system such as the speed of the constituents, whereas the momentum distribution does not give any information on the spatial location of the constituents. More complete information about the microscopic structure lies in the correlation between the momentum and coordinate spaces, i.e., to know where a particle is located and, at the same time, with what velocity it travels. This information is certainly attainable for a classical system for which one can define and study the phase-space distribution of the constituents. For a quantum mechanical particle, however, the notion of a phase-space distribution seems less useful because of the uncertainty principle. Nonetheless, the first phase-space distribution in quantum mechanics was introduced by Wigner in 1932 [1], and many similar distributions have been studied thereafter. These distributions have been used for various purposes in very diverse areas such heavy-ion collisions, quantum molecular dynamics, signal analysis, quantum information, optics, image processing, non-linear dynamics, etc.[2].

In this talk, we explore to what extent one can construct physically interesting and experimentally measurable phase-space distributions in quantum chromodynamics (QCD), and what information it contains about the QCD parton dynamics [3]. To facilitate the construction, we examine the uncertainty in the traditional interpretation of electromagnetic form factors due to relativity, and analyze the physical content of the Feynman parton distributions in the rest frame of the proton. We then introduce the phase-space Wigner distributions for the quarks and gluons in the proton, which contain most general one-body information of partons, corresponding to the full one-body density matrix in technical terms. After integrating over the spatial coordinates, one recovers the familiar transverse-momentum dependent parton distributions [4]. On the other hand, some reduced version of the distributions is related, through a specific Fourier transformation, to the generalized parton distributions (GPDs) which have been studied extensively in the literature in recent years [5]. Roughly speaking, a GPD is a one-body matrix element which combines the kinematics of both elastic form factors and Feynman parton distributions, and is measurable in hard exclusive processes. Therefore, the notion of phase-space distribution provides a new 3D interpretation of the GPDs in the rest frame of the proton.

Other interpretations of the GPD in the literature have been made in the IMF and impact parameter space [6].

2 Relativity constraint on interpretation of form factors and parton distributions

The electromagnetic form factors are among the first measured and mostly studied observables of the proton. They are defined as the matrix elements of the electromagnetic current between the proton states of different four-momenta. Because the proton is a spin one-half particle, the matrix element defines two form factors,

$$\langle p_2|j_\mu(0)|p_1\rangle=\bar{U}(p_2)\left\{F_1(q^2)\gamma_\mu+F_2(q^2)\frac{i\sigma_{\mu\nu}q_\nu}{2M_N}\right\}U(p_1)\,, \tag{1}$$

where F_1 and F_2 are the well-known Dirac and Pauli form factors, respectively, depending on the momentum transfer $q = p_2 - p_1$, and $U(p)$ is proton spinor normalized as $\overline{U}(p)U(p) = 2M_N$.

Since the beginning, it has been known that the physical interpretation of the proton form factors is complicated by relativistic effects [7]. Consider a system of size R and mass M. In relativistic quantum theory, the system cannot be localized to a precision better than its Compton wavelength $1/M$. Any attempt to do this with an external potential will result in creation of particle-antiparticle pairs. As a consequence, the static size of the system cannot be defined to a precision better than $1/M$. If $R \gg 1/M$, which is the case for all non-relativistic systems, the above is not a significant constraint. One can probe the internal structure of the system with a wavelength $(1/|\mathbf{q}|)$ comparable to or even much smaller than R, but still large enough compared to $1/M$ so that the probe does not induce an appreciable recoil. A familiar example is the hydrogen atom for which $RM_H \sim M_H/(m_e\alpha_{\rm em}) \sim 10^5$, and the form factor can be measured through electron scattering with momentum transfer $|\mathbf{q}| \ll M_H$.

When the probing wavelength is comparable to $1/M$, the form factors are no longer determined by the internal structure alone. They contain also the dynamical effects of Lorentz boosts because the initial and final protons have different momenta. In relativistic quantum theory, the boost operators involve nontrivial dynamical effects which result in the proton wave function being different in different frame (in the usual instant form of quantization). Therefore in the region $|\mathbf{q}| \sim M$, the physical interpretation of the form factors is complicated because of the entanglement of the internal and the center-of-mass motions in relativistic dynamics. In the limit $|\mathbf{q}| \gg M$, the former factors depend almost entirely on the physical mechanism producing the overall change of the proton momentum. The structural effect involved is a very small part of the proton wave function (usually the minimal Fock component only).

For the proton, $M_NR_N \sim 4$. Although much less certain than in the case of the hydrogen atom, it seems still sensible to have a rest-frame picture in terms of the electromagnetic form factors, so long as one keeps in mind that equally justified definitions of the proton sizes can differ by $\sim 1/M_N(R_NM_N)$. For example, the traditional definition of the proton charge radius in terms of the slope of the Sachs form factor $G_E(q^2)$ is 0.86 fm [8]. On the other hand, if one uses the slope of the Dirac form factor F_1 to define the charge radius, one gets 0.79 fm, about 10% smaller.

Since relativity makes the interpretation of the electromagnetic form factors non-unique, the best one can do is to choose one particular interpretation and work consistently. For example, when extracting the proton charge radius from the Lamb shift measurements, one shall use the same definition as from the electric form factor. The most frequently-used definition is that of Sachs [8], but other schemes are equally good and the scheme dependence disappears in the limit $MR \to \infty$. This is very much like the renormalization scheme dependence of parton densities due to radiative corrections at finite strong coupling constant $\alpha_{\rm s}$. In this paper, we adopt the Sachs interpretation of the form factors, which means that *the spatial distributions are defined to be the Fourier transformation of the Breit frame matrix elements.*

Parton distributions were introduced by Feynman to describe deep-inelastic scattering. They have the simplest interpretation in the IMF as the densities of partons in the longitudinal momentum x. To construct the quantum phase-space distributions for the quarks, we need an interpretation of the Feynman densities in the rest frame. This is because the IMF involves a Lorentz boost along the z-direction which destroys the rotational symmetry of the 3D space.

The physics of the Feynman quark distribution in the rest frame is made more clear through the notion of the *spectral function* (gauge-link omitted)

$$S(k) = \frac{1}{2p^+}\int d^4\xi {\rm e}^{ik\cdot\xi}\langle p|\overline{\Psi}(0)\gamma^+\Psi(\xi)|p\rangle\ . \tag{2}$$

which is the dispersive part of the single-quark Green's function in the proton. The physical meaning of $S(k)$ can be seen from its spectral representation,

$$\begin{aligned}S(k) &= \sum_n (2\pi)^4\delta^{(4)}(p-k-p_n)\langle p|\overline{\Psi}_k|n\rangle\gamma^+\langle n|\Psi(0)|p\rangle/2p^+ \\ &\sim \sum_n (2\pi)^4\delta^{(4)}(p-k-p_n)|\langle n|\Psi_{k+}|p\rangle|^2 \end{aligned} \tag{3}$$

where Ψ_k is a Fourier transformation of quark field $\Psi(\xi)$: It is the probability of annihilating a quark (or creating an antiquark) of *four*-momentum k (three-momentum $\mathbf{k}$ and the off-shell energy $E = k^0$) in the proton, leading to an "on-shell" state of energy-momentum $p_n = p - k$. The quark here is off-shell because if p_n and p are both "on-shell", $k^2 \neq m_q^2$ in general. [That the partons are off-shell are in fact also true in the IMF calculations.] Of course, in QCD $|n\rangle$ is not in the Hilbert space, but the spectral function itself is still a meaningful quantity.

Since the quarks are ultra-relativistic, Ψ_k contains both quark and antiquark Fock operators. One cannot in general separate quark and anti-quark contributions, unlike in the non-relativistic systems in which only the particle or antiparticle contribute. In fact, if one expands the above expression, one finds pair creations and annihilation terms. However, this is also true for the usual charge density. Therefore we can speak of $S(k)$ as a distribution of vector charges and currents, but not a particle density. In nuclear physics where the non-relativistic dynamics dominates, the proton spectral function in the nucleus is positive definite and can be regarded as a particle density. The nuclear spectral function is directly measurable through pick-up and knock-out experiments, in which E and $\mathbf{k}$ are called the missing energy and missing momentum, respectively (see for example [9]).

It is now easy to see that in the rest frame of the proton, the Feynman quark distribution is

$$q(x) = \sqrt{2} \int \frac{d^4k}{(2\pi)^4} \delta(k^0 + k^z - xM_N) S(k) \ . \qquad (4)$$

The x variable is simply a special combination of the off-shell energy k^0 and momentum k^z. The parton distribution is the spectral function of quarks projected along a special direction in the four-dimensional energy-momentum space. The quarks with different k^0 and k^z can have the same x, and moreover, the both $x > 0$ and $x < 0$ distributions contain contributions from quarks and anti-quarks.

3 Quantum phase-space distributions

Suppose we have a one-dimensional quantum mechanical system with wave function $\psi(x)$, the Wigner distribution is defined as

$$W(x,p) = \int d\eta e^{ip\eta} \psi^*(x - \eta/2)\psi(x + \eta/2) \ , \qquad (5)$$

where we have set $\hbar = 1$. When integrating out the coordinate x, one gets the momentum density $|\psi(p)|^2$, which is positive definite. When integrating out p, the positive-definite coordinate space density $|\psi(x)|^2$ follows. For arbitrary p and x, the Wigner distribution is not positive definite and does not have a probability interpretation. Nonetheless, for calculating the physical observables, one can just take averages over the phase-space as if it is a classical distribution

$$\langle \hat{O}(x,p) \rangle = \int dx dp \, W(x,p) O(x,p) \qquad (6)$$

where the operators are ordered according to the Weyl association rule. For a single-particle system, the Wigner distribution contains everything there is in the quantum wave function. For a many-body system, the Wigner distribution can be used to calculate the averages of all one-body operators. Sign changes in the phase-space are a hint that it carries non-trivial quantum phase information.

In QCD, the single-particle wave function must be replaced by (gauge-invariant) quantum fields, and hence it is natural to introduce the *Wigner operator*,

$$\hat{\mathcal{W}}_\Gamma(\mathbf{r}, k) = \int d^4\eta e^{ik\cdot\eta} \overline{\Psi}(\mathbf{r} - \eta/2) \Gamma \Psi(\mathbf{r} + \eta/2) \ , \qquad (7)$$

where $\mathbf{r}$ is the quark phase-space position and k the phase-space four-momentum conjugated to the spacetime separation η. Γ is a Dirac matrix defining the types of quark densities because the quarks are spin-1/2 relativistic particles. Depending on the choice of Γ, we can have vector, axial vector, or tensor density.

For non-relativistic systems for which the center-of-mass is well-defined and fixed, one can define the phase-space distributions by taking the expectation value of the above Wigner operators in the $\mathbf{R} = 0$ state. For the proton for which the recoil effect cannot be neglected, the rest-frame state cannot be uniquely defined. Here we follow Sachs, defining a rest-frame matrix element as that in the Breit frame, averaging over all possible 3-momentum transfers. Therefore, we construct the quantum phase-space quark distribution in the proton as,

$$\begin{aligned} W_\Gamma(\mathbf{r}, k) &= \frac{1}{2M_N} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left\langle \mathbf{q}/2 \left| \hat{\mathcal{W}}_\Gamma(\mathbf{r}, k) \right| - \mathbf{q}/2 \right\rangle \qquad (8) \\ &= \frac{1}{2M_N} \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \left\langle \mathbf{q}/2 \left| \hat{\mathcal{W}}_\Gamma(0, k) \right| - \mathbf{q}/2 \right\rangle \ , \end{aligned}$$

where the plane-wave states are normalized relativistically. The most general phase-space distribution depends on *seven* independent variables.

The only way we know how to probe the single-particle distributions is through high-energy processes, in which the light-cone energy $k^- = (k^0 - k^z)/\sqrt{2}$ is difficult to measure, where the z-axis refers to the momentum direction of a probe. Moreover, the leading observables in these processes are associated with the "good" components of the quark (gluon) fields in the sense of light-cone quantization [10], which can be selected by $\Gamma = \gamma^+$, $\gamma^+\gamma_5$, or $\sigma^{+\perp}$ where $\gamma^+ = (\gamma^0 + \gamma^z)/\sqrt{2}$. The direction of the gauge link, n^μ, is then determined by the trajectories of high-energy partons traveling along the light-cone $(1, 0, 0, -1)$ [11,12]. Therefore, from now on, we restrict ourselves to the reduced Wigner distributions by integrating out k^-,

$$W_\Gamma(\mathbf{r}, \mathbf{k}) = \int \frac{dk^-}{(2\pi)^2} W_\Gamma(\mathbf{r}, k) \ , \qquad (9)$$

with a light-cone gauge link is now implied. Unfortunately, there is no known experiment at present capable of measuring this 6-dimensional distribution which may be called the *master* or *mother* distribution.

Further phase-space reductions lead to measurable quantities. Integrating out the transverse momentum of partons, we obtain a 4-dimensional quantum distribution

$$\begin{aligned} \tilde{f}_\Gamma(\mathbf{r}, k^+) = \frac{1}{2M_N} \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \int \frac{d\eta^-}{2\pi} e^{i\eta^- k^+} \\ \times \left\langle \mathbf{q}/2 \left| \overline{\Psi}(-\eta^-/2) \Gamma \Psi(\eta^-/2) \right| - \mathbf{q}/2 \right\rangle \ . \ (10) \end{aligned}$$

The matrix element under the integrals is what defines the GPDs. More precisely, if one replaces k^+ by Feynman variable xp^+ ($p^+ = E_q/\sqrt{2}$, proton energy $E_q = \sqrt{M^2 + \mathbf{q}^2/4}$) and η^- by λ/p^+, the reduced Wigner distribution becomes the Fourier transformation of the GPD $F_\Gamma(x, \xi, t)$

$$f_\Gamma(\mathbf{r},x) = \frac{1}{2M_N}\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} F_\Gamma(x,\xi,t) \ . \qquad (11)$$

In the present context, the relation between kinematic variables are $\xi = q^z/(2E_q)$ and $t = -\mathbf{q}^{\,2}$. Taking $\Gamma = \sqrt{2}\gamma^+$, F_{γ^+} is the same as that in [13]

$$F_{\gamma^+}(x,\xi,t) = \frac{d\lambda}{2\pi} e^{i\lambda x} \left\langle \mathbf{q}/2 \left| \overline{\psi}(-\lambda n/2)\mathcal{L}\sqrt{2}\gamma^+\psi(\lambda n/2)\right| -\mathbf{q}/2\right\rangle \qquad (12)$$

which defines $H(x,\xi,t)$ and $E(x,\xi,t)$.

The phase-space function $f_{\gamma^+}(\mathbf{r},x)$ can be used to construct 3D images of the quarks for every selected Feynman momentum x in the rest frame of the proton. These images provide the pictures of the proton seen through the Feynman momentum (or "color" or x) filters. They also may be regarded as the result of a quantum phase-space tomography of the proton. We remind the reader again that the Feynman momentum in the rest-frame sense is a special combination of the off-shell energy and momentum along z, namely $E + k^z$. Integrating over the z coordinate, the GPDs are set to $\xi \sim q^z = 0$, and the resulting two-dimensional density $f_{\gamma^+}(\mathbf{r}_\perp, x)$ is just the impact-parameter-space distribution [6]. Further integration over $\mathbf{r}_\perp$ recovers the usual Feynman parton distribution.

The physical content of the above distribution is further revealed by examining its spin structure. Working out the matrix element in (12),

$$\frac{1}{2M_N}F_{\gamma^+}(x,\xi,t) = [H(x,\xi,t) - \tau E(x,\xi,t)] + i[\mathbf{s}\times\mathbf{q}]^z \frac{1}{2M_N}\left[H(x,\xi,t)+E(x,\xi,t)\right], \quad (13)$$

where $\tau = \mathbf{q}^2/4M_N^2$. The first term is independent of the proton spin, and is considered as the phase-space charge density

$$\rho_+(\mathbf{r},x) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}}[H(x,\xi,t) - \tau E(x,\xi,t)] \ . \quad (14)$$

The second term depends on the proton spin and can be regarded as the third component of the phase-space vector current $j_+^z(\mathbf{r},x) =$

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} i[\mathbf{s}\times\mathbf{q}]^z \frac{1}{2M_N}\left[H(x,\xi,t)+E(x,\xi,t)\right] \ . \quad (15)$$

The E-term generates a convection current due to the orbital angular momentum of massless quarks and vanishes when all quarks are in the s-orbit. The physics in separating f_γ^+ into ρ_+ and j_+^z can be seen from the Dirac matrix γ^+ selected by the high-energy probes, which is a combination of time and space components.

4 Quark charge distribution seen through Feynman-momentum (x) filters

Once the GPDs are extracted from experimental data or lattice QCD calculations, the phase-space charge/current distributions can be obtained by straightforward Fourier transformations. Without a first-hand knowledge on the GPDs at present, we may be able to learn some general features of the phase-space distributions form GPD models.

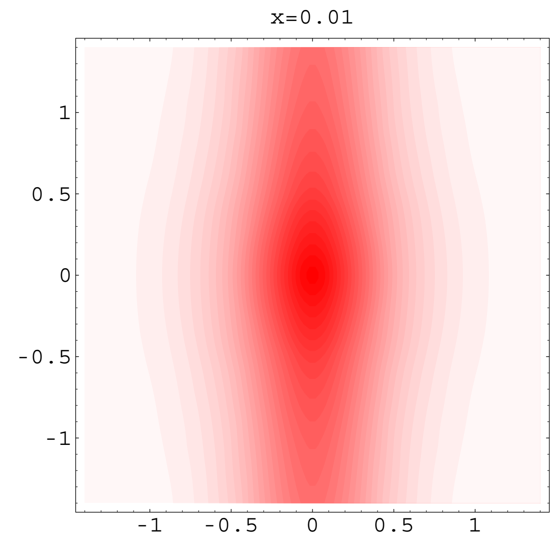

Fig. 1. The up-quark charge density in the proton when Feynman momentum is 0.01

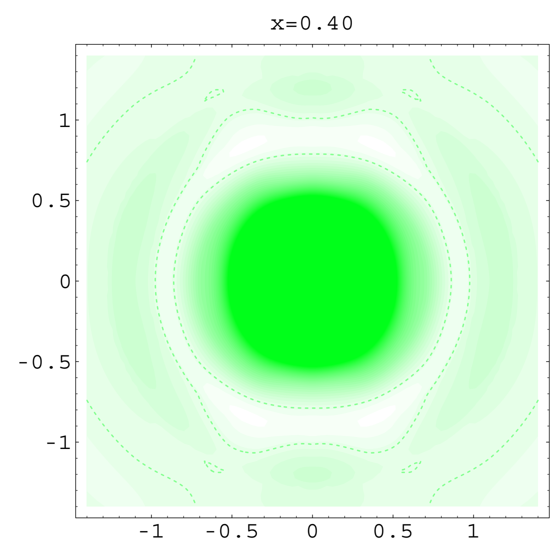

Fig. 2. Same as Fig. 1, the Feynman momentum is 0.4

The GPDs have been parametrized directly to satisfy various constraints, including 1) the first moments reducing to the measured form factors, 2) the forward limit reproducing the Feynman parton distributions, 3) the x-moments satisfying the polynomiality condition [5], and 4) the positivity conditions [14]. Here, we use a new parametrization without assuming factorized dependence on the t and other variables [3].

In Fig. 1, we show the up-quark charge distributions calculated from $H_u(x,\xi,t)$ for various values of $x = 0.01$. While the intensity of the plots indicates the magnitude of the positive distribution, the lighter areas below the

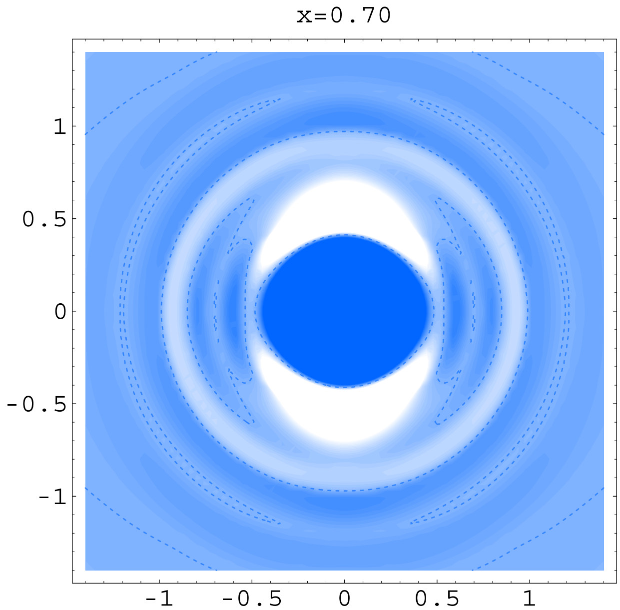

Fig. 3. Same as Fig. 1, the Feynman momentum is 0.7

ground-zero contours indicate negative values. The image is rotationally symmetric in the $\mathbf{r}_\perp$-plane (shown as the horizontal axis only). At small x, the distribution extends far beyond the nominal proton size along the vertical z direction. The physical explanation for this is that the position space uncertainty of the quarks is large when x is small, and therefore the quarks are de-localized along the longitudinal direction. This de-localization reflects a very peculiar part of the proton wave function and shows long-range correlations as verified in high-energy scattering. In a nucleus, the parton distributions at small x are strongly modified because of the spatial 1overlap between the protons. Figure 2 shows the charge density at $x = 0.4$ which is roughly round. At larger x, the momentum along z direction is of order proton mass, the quarks are localized to within $1/M_N$. The quantum mechanical nature of the distribution becomes distinct as there are significant changes in the sign at different spatial regions, shown in Fig. 3. There pictures provide a fantastic visualization of the quarks in the proton.

To summarize, we have introduced the concept of the quantum phase-space distributions for quarks and gluons in the proton. These distributions are measurable through their relations to transverse-momentum dependent parton distributions and generalized parton distributions. They can be used to visualize the phase-space motion of the quarks, and hence allow studying the contribution of the quark orbital angular momentum to the spin of the proton. This work was supported by the U. S. Department of Energy via grant DE-FG02-93ER-40762.

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